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ON THE ACCURACY OF THE GREEN–RIVLIN REPRESENTATION FOR VISCOELASTIC MATERIALS

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Abstract—The applicability of the Green–Rivlin representation for non-linear viscoelastic materials is investigated by taking the errors of the experimental data into account. Two bounds for the relative error of the relaxation or creep kernel functions have been obtained as a function of measurement errors. The analysis shows that the kernels are very sensitive to experimental errors. Moreover, when the number of terms in the integral expansion is increased, so is the error. The theoretical estimates are compared with the errors obtained when using experimental data of a Lexan-polycarbonate polymer. The comparison is satisfactory for one of the bounds, at least in this particular case.

1. INTRODUCTION

GREEN and Rivlin [1] have presented a theory for non-linear viscoelastic materials (simple materials with memory), in which the stress constitutive functional is expanded as an infinite series of multiple integrals. This representation has been used, e.g. by Ward and Onat [2], Hadley and Ward [3], Onaran and Findley [4], Lifshitz and Kolsky [5], Neis and Sackman [6], to describe the non-linear mechanical response of polymers. Lockett [7] has discussed the type of experiments that are needed in order to characterize a constitutive equation involving multiple integrals up to and including a third order term.

It has been generally assumed that the accuracy of this method can be obtained, at least in principle, by increasing the number of terms. We will prove, using perturbation theory, that this assertion is not generally true since the kernels of the multiple integrals turn out to be very sensitive to scatter in the experimental data. This sensitivity is increased as the number of terms increases.

Even when the data is smoothed in the manner suggested in [5] and [6], the system of linear equations that are used to evaluate the kernel functions remains ill-conditioned. Therefore large errors are to be expected for materials of high order.

2. ERROR ANALYSIS

We consider, for the sake of simplicity, a uniaxial state of stress. Hence, the Green–Rivlin representation of the stress functional for a material of order‡ n reduces to

$$\sigma(t) = \sum_{k=1}^{n} S_k, \tag{1}$$

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‡ The order of the material is equal to the largest number n of the iterated integrations occurring in (1).

where

$$S_k = \int_{-\infty}^{t} \int_{-\infty}^{t} \dots \int_{-\infty}^{t} E_k(t-\tau_1, t-\tau_2 \dots t-\tau_k) \frac{\partial e(\tau_1)}{\partial \tau_1} \frac{\partial e(\tau_2)}{\partial \tau_2} \dots \frac{\partial e(\tau_k)}{\partial \tau_k} d\tau_1 \, d\tau_2 \dots d\tau_k,$$

$\sigma(t)$ and $e(t) = \varepsilon(t) + \varepsilon^2(t)/2$ are respectively the stress and finite strain in the X-direction, $\varepsilon(t) = \partial u/\partial X$ is the infinitesimal strain and E_k represents the relaxation kernel functions to be determined experimentally. The strain functional that relates $e(t)$ to the history of the stress $\sigma(\tau)$ is given by an expansion similar to (1).

Now let us consider n simple relaxation tests under a constant strain ${}_0e_i$ of the form

$$e_i(t) = {}_0e_i H(t) \qquad i = 1, 2 \dots n, \tag{2}$$

where H denotes the Heaviside step function

$$H = 0 \qquad t < 0, \qquad H = 1 \qquad t \geq 0.$$

Substitution of (2) into (1) yields the stress response to this program:

$$\sigma_i = \sum_{j=1}^{n} ({}_0e_i)^j E_j \qquad i = 1, 2 \dots n, \tag{3}$$

where the notation

$$\sigma_i = \sigma(t_k, {}_0e_i), \qquad E_j = E_j(t_k, t_k \dots t_k)$$

is used and t_k denotes the instant at which the measurements are made. Thus, the program (2) provides the necessary equations for the determination of the values of the unknown kernels $E_1, E_2 \dots E_n$ along the line $\tau_1 = \tau_2 = \dots = \tau_n = t_k$.

If the first equation (3) is divided by ${}_0e_i$, the second by ${}_0e_2 \dots$ and the last one by ${}_0e_n$, the equation (3) can be written in the form

$$\mathbf{A}\mathbf{x} = \mathbf{t}, \tag{4}$$

where the elements of the matrix $\mathbf{A}$ are $a_{ij} = ({}_0e_i)^{j-1}$ and the elements of the vectors $\mathbf{x}$ and $\mathbf{t}$ are respectively equal to

$$x_i = E_i, \qquad t_i = \sigma_i / {}_0e_i.$$

Note that

$$\det \mathbf{A} = \begin{vmatrix} 1 & {}_0e_1 & ({}_0e_i)^2 & \dots & ({}_0e_1)^{n-1} \\ 1 & {}_0e_2 & ({}_0e_2)^2 & \dots & ({}_0e_2)^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & {}_0e_n & ({}_0e_n)^2 & \dots & ({}_0e_n)^{n-1} \end{vmatrix} \tag{5}$$

is a Vandermode determinant† and it is always different from zero for ${}_0e_i \neq {}_0e_j$ $(i \neq j)$. This assures the existence of a unique solution of (4).

Now let us examine (4). First, if the behavior of the material does not depart too much from linearity, the vector $\mathbf{t}$ is nearly proportional to the first column of $\mathbf{A}$. Second, the matrix $\mathbf{A}$ may be ill-conditioned since it is not diagonally dominant and its elements could be of different orders of magnitude. Consequently one should expect computational difficulties when solving (4) for $\mathbf{x}$, especially for the determination of $E_2, E_3 \dots E_n$.

† This interesting property of $\mathbf{A}$ has apparently been ignored in the literature mentioned in Section 1.

To make this assertion more precise, consider that the measured values of the stresses σ_i and strains ${}_0e_i$ are affected by experimental errors $\delta\sigma_i$ and δe_i respectively, so that $\mathbf{t}$ and $\mathbf{A}$ will be affected by the errors* $\delta\mathbf{t}$ and $\delta\mathbf{A}$. Hence the solution $\mathbf{x}$ of (4) will also be affected by an error $\delta\mathbf{x}$ since one is solving the system of linear equations†

$$(\mathbf{A}+\delta\mathbf{A})(\mathbf{x}+\delta\mathbf{x}) = \mathbf{t}+\delta\mathbf{t} \tag{6}$$

instead of (4). Note that $\delta\mathbf{t} = \delta\mathbf{t}(\delta\sigma_i, \delta e_i)$, $\delta\mathbf{A} = \delta\mathbf{A}(\delta e_i)$. By subtracting (4) from (6) and rearranging one gets

$$(\mathbf{I}+\mathbf{A}^{-1}\delta\mathbf{A})\delta\mathbf{x} = \mathbf{A}^{-1}\delta\mathbf{t}-\mathbf{A}^{-1}\delta\mathbf{A}\mathbf{x} \tag{7}$$

where $\mathbf{I}$ is the unity matrix. Since $\mathbf{I}+\mathbf{A}^{-1}\delta\mathbf{A}$ is invertible provided $\|\mathbf{A}^{-1}\delta\mathbf{A}\| < 1$, where $\|\cdot\|$ indicates a suitable matrix norm,† it follows that

$$\|(\mathbf{I}+\mathbf{A}^{-1}\delta\mathbf{A})^{-1}\| \leq (1-\|\mathbf{A}^{-1}\delta\mathbf{A}\|)^{-1} \quad \text{if } \|\mathbf{I}\| = 1,$$

and the relative error $\|\delta\mathbf{x}\|/\|\mathbf{x}\|$ is expressed by

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{\kappa}{(1-\kappa\|\delta\mathbf{A}\|/\|\mathbf{A}\|)}\left(\frac{\|\delta\mathbf{t}\|}{\|\mathbf{t}\|}+\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}\right), \tag{8}$$

where $\kappa = \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\|$ is the condition number of the matrix $\mathbf{A}$. The matrix and vector norms that will be used are

$$\|\mathbf{x}\| = \max_i |x_i|, \qquad \|\mathbf{A}\| = \max_i \sum_{j=1}^{n} |a_{ij}|, \tag{9}$$

which verify the requirements $\|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\|\,\|\mathbf{x}\|$, $\|\mathbf{I}\| = 1$ used above. In general, the sensitivity of the system (4) will depend on the condition number κ. When $\kappa \gg 1$, (8) can overestimate the error.‡

If $\delta\mathbf{A} \equiv 0$, the formula

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}^{-1}\|\frac{\|\delta\mathbf{t}\|}{\|\mathbf{x}\|} \tag{10}$$

gives a better estimate of the relative error.

For example, let us consider a material of order three ($n = 3$) and assume that

$$|{}_0e_i| = i\mu \qquad i = 1, 2, 3, \tag{11}$$

where μ is a positive number.

It immediately follows that

$$\|\mathbf{A}\| = 1+3\mu+9\mu^2, \qquad \|\mathbf{A}^{-1}\| = \max\left(7, \frac{8}{\mu}, \frac{2}{\mu^2}\right), \tag{12}$$

where the notation max(., ., .) indicates that the maximum number between the parentheses has to be taken. Consequently, for any value of $\mu \neq 0$, $\kappa \gg 1$. If a material of order four

† If rounding errors are also considered, they may be added to $\delta\mathbf{t}$ and $\delta\mathbf{A}$.

‡ See e.g. Wilkinson [8].

under the same step loading (11) for $i = 1, 2, 3, 4$ is considered, an easy computation shows that

$$\|\mathbf{A}\| = 1+4\mu+16\mu^2+64\mu^3,$$
$$\|\mathbf{A}^{-1}\| = \max\left(15, \frac{68}{3\mu}, \frac{10}{\mu^2}, \frac{7}{6\mu^3}\right). \tag{13}$$

Finally the condition number κ of a material of order n under the loading (11) is obtained by using the norms

$$\|\mathbf{A}\| = \sum_{i=1}^{n} (i\mu)^{i-1}, \qquad \|\mathbf{A}^{-1}\| = \max\left(B_1, \frac{B_2}{\mu}, \ldots \frac{B_n}{\mu^{n-1}}\right) \tag{14}$$

which have been obtained by induction. B_i are suitable positive constants.

The above results show that the value of the kernels at $\tau_1 = \tau_2 = \ldots = \tau_n = t_k$ are very sensitive to errors in the experimental values of σ_i and ${}_0e_i$. Moreover, if the number of terms in the integral expansion (1) is increased, so is the value of κ. Thus, the experimental accuracy has to be further improved in order to keep $\|\delta\mathbf{x}\|/\|\mathbf{x}\|$ within a certain limit.

When the constant strains ${}_0e_i$ are arbitrarily selected rather than given by (11), it will be necessary to compute $\mathbf{A}^{-1}$ in order to apply either of the estimates (8) and (10).

Let us check the usefulness of this error analysis when applied to experimental data: stress relaxation experiments† of the form (2) performed on a Lexan polycarbonate polymer at a temperature of 25°C. Table 1 shows the different uniaxial strains ${}_0e_i$, the corresponding stresses σ_i and the computed values of E_i at the instant $t_k = 1000$ sec. It has been assumed that the polymer behaves like a material of order three.

The calculations give

$$\|\mathbf{A}\| = 1{\cdot}045, \qquad \|\mathbf{A}^{-1}\| = 11{,}765, \qquad \kappa = 12{,}294.$$

If only errors in the measured stresses are considered as in Table 1, one obtains by solving (6) the new values of the kernels $E_i' = E_i + \delta E_i$ as shown in Table 1. The estimates (8) and (10) give the following results

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq 221 \qquad \text{(estimate (8))},$$

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq 3{\cdot}97 \qquad \text{(estimate (10))},$$

TABLE 1

i	${}_0e_i$	σ_i ($10^8 x$ dynes/cm^2)	$\delta\sigma_i/\sigma_i$	E_i ($10^{10} x$ dynes/cm^2)	E_i'
1	0·015	2.21	0·014	1·602	1·548
2	0·035	4·39	0·018	−7·575	−0·5201
3	0·0435	4·98	−0·009	−67·54	−206·7

† The author is indebted to Professor I. Yannas of M.I.T. for providing unpublished experimental data.

whereas the computed value is

$$\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|} = 2{\cdot}06.$$

This example shows that small errors in the stresses become greatly amplified in the kernel values. Although in this case (8) is very pessimistic, the estimate (10) is in agreement with the computed value. Calculations performed using different error vectors $\delta \mathbf{t}$ have shown a similar tendency.

3. CONCLUDING REMARKS

The analysis of Section 2 is obviously applicable to the creep kernel functions of the strain functional. Besides, it is unnecessary to consider more complex loading programs once it has been proved that the value of the kernels along the lines $\tau_1 = \tau_2 = \ldots = t_k$ are very sensitive to experimental errors. A similar comment can be made for three-dimensional loading programs.

In conclusion, the error analysis performed in this work has shown that the use of the Green–Rivlin representation with a large number of terms would require a level of experimental accuracy which is apparently unavailable at the present time. For a fixed experimental accuracy, the order of the material cannot be increased indefinitely because of the numerical errors that may arise in the computation of the kernels. Thus, other methods should be used, e.g. the one proposed by Pipkin and Rogers [9].

Finally it is to be noted that these results are also applicable to the determination of the kernels of non-linear functionals used to characterize retarded non-linear responses to time-dependent external excitations (Nakada [10]).

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Абстракт—Исследуется применимость представления Грина-Ривлина для нелинейных, вязко-упругих материалов, путем учета поярешностей, вытекающих из экспериментальных данных. Получаются два ограничения для действительной погрешности функций реляксации или функции ползучести в виде функции погрешностей измерения. Анализ указывает на болъшую чувствительность ядер функции в зависимости от экспериментальных погрешностей. Кроме того, увеличение числа выражений интегралъного разложения также ошибочно. Сравниваются теоретические исследования с погрешностями, полученными при полъзовании экспериментальных данных для много-карбонатного полимера Лексана. Сравнение удовлетворителъно для одного из ограничений, по крайной мере дл. этого особого случая.